Exercise 73

The equation $x^2 - xy + y^2 = 3$ represents a "rotated ellipse," that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.

Solution

In order to find where the ellipse crosses the x-axis, set $y = 0$ and solve the equation for x.

$$x^2 - x(0) + (0)^2 = 3$$

$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx}(3)$$

Use the chain rule to differentiate $y = y(x)$.

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(3)$$

$$2x - \left[\frac{d}{dx}(x)\right] y - x\left[\frac{d}{dx}(y)\right] + 2y\frac{dy}{dx} = 0$$

$$2x - (1)y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Solve for dy/dx.

$$2x - y + (-x + 2y)\frac{dy}{dx} = 0$$

$$(-x + 2y)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{-x + 2y}$$

At the x-intercept $(-\sqrt{3}, 0)$, the slope of the tangent line is

$$\frac{dy}{dx} = \frac{(0) - 2(-\sqrt{3})}{-(-\sqrt{3}) + 2(0)} = 2.$$

At the x-intercept $(\sqrt{3}, 0)$, the slope of the tangent line is

$$\frac{dy}{dx} = \frac{(0) - 2(\sqrt{3})}{-(\sqrt{3}) + 2(0)} = 2.$$

Therefore, the tangent lines at these points are parallel, both with slope 2.

The graph below shows that the tangent lines to the curve at the x-intercepts are parallel.